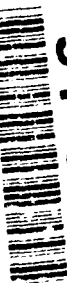AD-A236 904

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DIMENSIONAL ANALYSIS IN MATHEMATICAL
MODELING SYSTEMS
A SIMPLE NUMERICAL METHOD

Hemant K. Bhargava

February 1991

NAVAL POSTGRADUATE SCHOOL
Monterey, California

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Dimensional Analysis in Mathematical Modeling Systems A Simple Numerical Method

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Abstract

This paper discusses dimensional manipulation, essentially a problem requiring symbolic mathematics techniques, using a numerical approach. The numerical method obeys the laws of dimensional arithmetic. This is achieved by specifying an encoding of units of measurement as prime numbers, and manipulating the resulting expressions numerically. The unique factorization theorem is applied to show that this method makes trivial the problems of dimensional simplification and verification of dimensional equivalency, which are central issues in dimensional arithmetic. The solution has immediate application in mathematical modeling systems, chiefly in the model validation and model solution phases.

1

1 Introduction

This paper presents an efficient and simple numerical method for dimensional arithmetic. The problem of dimensional manipulation can be viewed as one of symbolic mathematics [25, 26, 17], since dimensions (quantities, units of measurement, see §2) are non-numeric symbols. However, we transform it to a simple numerical problem, and develop an algorithm for this transformed problem. There are three key steps in this approach. First, we recognize the special nature of the laws of dimensional arithmetic. Second, we develop a prime-encoding of dimensions, in which each unit of measurement is represented by a prime number. Third, we apply the unique factorization theorem from number theory to show that numeric arithmetic applied to this prime-encoding obeys the laws of dimensional arithmetic.

Dimensional arithmetic, or the calculus of dimensions, involves operations on dimensions analogous to the arithmetic operations on numbers. The techniques required to perform dimensional arithmetic as a symbolic mathematics problem are implemented in several computer algebra programs (e.g., Macsyma [19], Reduce [20], and Mathematica [29]). For example, such systems can prove that $(a^2 + ab + b^2 + ba) = (a + b)^2$. With some effort, since the laws of physical algebra are a minor variant on those of standard arithmetic performed on numbers [23], dimensional arithmetic can be, and has been, performed using these systems. Our alternative, numerical, method does not require specialized symbolic manipulation techniques, and has been used in implementing features for dimensional analysis in a model management system TEFA, described elsewhere [2].

A number of languages and systems have been developed for mathematical (particularly, for mathematical programming) modeling; some of these

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are AMPL [12], GAMS [5], SML [13], FW/SM [14], ANALYZE [15], TEFA [2], LINGO [8], and LPL [18]). This paper was motivated by the need to represent and manipulate dimensional information [6, 3] in such languages and systems. Dimensional analysis has several potential applications in mathematical modeling systems [2, 6, 9]. Transformations of units of measurement are required in model solution and model integration. Dimensional simplification and verification of dimensional consistency of expressions is useful in model formulation and model validation. While transformation of units is straightforward and is supported in various modeling systems, database systems, and symbolic mathematics systems, automatic dimensional consistency checking and dimensional simplification is found in only a few symbolic mathematics systems (e.g., Macsyma, Mathematica).

Of course, the concepts of dimensions and the usefulness of dimensional analysis have been known for a long time. Fourier is credited with establishing the principle of dimensional homogeneity in the 1820s [22], and Buckingham's *Pi Theorem* for the identification of dimensionless groups of variables [7] has been applied in dimensional analysis for several decades.[1] Dimensional analysis is used in high school physics courses for the derivation and verification of the laws of nature, and has found recent applications in qualitative physics [4]. Sedov [24] discusses dimensions and units with a number of examples. Wallot [28] presents various dialects for units and dimensions, and discusses the algebraic and arithmetic relationships between systems of units. Massey [23] provides an excellent practical guide to currently accepted units of measurement and conversion factors within and across systems of measurement. The usefulness of dimensional analysis is well recognized and exploited in the physical sciences, and in scientific computing systems. Such

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[1]See [17] for a computer solution of the theorem.

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is not the case, however, in management science, and in many model management systems proposed for management science modeling.[2] We aim to bridge that gap by presenting a simple numerical method that can be incorporated in to virtually any modeling system (as has been done in TEFA), database system, or programming language for the purposes of verifying dimensional consistency. Further, since many modeling languages do not discuss the representation of dimensional information, we also specify a formal language for representation of dimensional information, which language can be embedded in most modeling languages (as, again, has been done in the language of TEFA).

The rest of this paper is organized as follows. We briefly discuss the concepts of dimensions, units and the laws of dimensional consistency and arithmetic in §2. We present our method for dimensional arithmetic in §3, and develop a logic-based language for representing dimensional information in §4. We conclude with a discussion of the applications and limitations of our approach (§5).

2 Dimensions and Units of Measurement

The dimension of a numeric-valued modeling variable is defined in terms of the **quantity** it measures, and its **unit of measurement**. Each quantity has a **base-unit** of measurement, and several other units, in each of a few widely accepted **systems of measurement**. These units are related by laws of conversion within and across systems of measurement. In the International Metric System, or "Systéme Internationale d'Unites" (**SI**), there are seven

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[2]In fact, studies of modeling practise often reveal examples of models with incorrect and dimensionally inconsistent expressions.

fundamental[3] quantities and base units [1, 23], shown below, and several
other derived quantities and units.

UNITS FOR THE SI SYSTEM OF MEASURES

Quantity Measured	Base Unit	Other Units
Length	Meter (m)	Kilometer, Centimeter
Mass	Kilogram (kg)	Gram, Milligram
Time	Second (s)	Hour, Minute
Electric Current	Ampere (A)	
Temperature	°Kelvin (°K)	°Celsius, °Fahrenheit
Luminous Intensity	Candela (cd)	Candle-power
Amount of Substance	Mole (mol)	Kilomole

There are also dimension*less* quantities, which have no units (it is useful
to think of this unit as the number 1). In most applications of manage-
ment science modeling it is useful to enrich this vocabulary by introducing
other quantities and units. For example, the quantity money is measured
in dollars, yen, and so on. Other examples of units are barrels-of-oil.
truck-loads, 1985-dollars. The solution we propose is general enough to
apply to such enhanced vocabularies. (It is also unaffected by the choice of
a base unit.) For the purpose of this paper, it is sufficient to consider units
of measurement (and to ignore quantities). This is because 1) given a vari-
able's units, we can easily infer the quantity it measures, and 2) information
about units is necessary to verify dimensional consistency of a mathematical
expression. Hence, in the rest of the paper, we will use the words dimension,
dimensional and so on. to refer to units of measurement. However, we need
to make a distinction between fundamental units, and derived units. A

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[3]These are fundamental in the sense that they form a *basis* for the group of all quantities
and units respectively. There are several bases, but the selection of fundamental quantities
and base units is governed also by convention and convenience.

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fundamental unit can not be written as the product of two or more fundamental units or their inverses. For example, (using abbreviations) **m**, **km**, **s**, and **kg**, are some fundamental units. A derived unit is obtained as a product of two or more fundamental units or their inverses. For example, kgm^2/s^3 is a derived unit for Power. A **unit** is a fundamental unit or a derived unit.

2.1 Laws of dimensional consistency

The laws for obtaining dimensionally consistent (d.c.) expressions are stated below.[4]

1. Two functional expressions may be added or subtracted only if they are dimensionally equivalent (d.e.). (Expressions of the form $\phi + \psi$, $\phi - \psi$ are d.c. iff ϕ and ψ are d.e.)

2. Two functional expressions may be compared for equality or inequality (resulting in a conditional expression) only if their dimensions are equivalent. (Expressions of the form $\phi = \psi$, $\phi < \psi$, $\phi > \psi$, $\phi \leq \psi$, and $\phi \geq \psi$ are d.c. iff ϕ and ψ are d.e.)

3. Two functional expressions may be multiplied irrespective of their dimensions. (Expressions of the form $\phi * \psi$ are d.c.)

4. A functional expression can be reciprocated irrespective of its dimension. (Expressions of the form $1/\phi$ are d.c.)

5. The exponent of a functional expression must be dimensionless. (Expressions of the form ϕ^ψ are d.c. only if ψ is dimensionless.)

6. The exponent of a functional expression can be fractional only if a) each fundamental unit in the functional expression has a power that is a multiple of the inverse of that fraction, or if b) the functional expression is dimensionless. (Expressions of the form ϕ^ψ are d.c. if the fundamental units of ϕ^ψ have integer powers, i.e., ψ is an integer, or if ϕ is dimensionless, or if each fundamental unit in ϕ has a power that is a multiple of $1/\psi$.)

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[4]The last three laws (5,6,7) might seem to be unreasonable; we will have more to say about that in §5. These laws are consistent with observations by several authors about physical systems and rules for using dimensions [10, 23, 27].

7. Functions which can be expressed as power series (e.g., trignometric functions, hyperbolic functions) can be applied only to dimensionless expressions.

These laws can be used for dimensional validation of expressions. Note that such validation requires verification of dimensional equality.

2.2 Laws of dimensional arithmetic

The laws of dimensional arithmetic are slightly different from that of standard arithmetic, and this difference makes our proposed solution work.

1. The dimension of the sum (or difference) of two expressions is the same as the dimension of either of them if the two expressions have equivalent dimensions. Otherwise, it is not defined (see laws for dimensional consistency).

2. The dimension of the product (quotient) of two expressions is the product (quotient) of the dimensions of the two expressions. A dimensionless expression has dimension u_0 $(= 1)$ which is an identity for dimensional multiplication.

3. The dimension of the exponent of an expression is the exponent of its dimension.

4. Any function of an expression with dimension u_0 yields an expression of dimension u_0.

Note that we still have the problem of establishing dimensional equivalence (in Law 1), and of performing dimensional simplification (in Law 2).

3   A Numerical Approach to Dimensional Arithmetic

We will first restate the laws of dimensional arithmetic in terms of the operators for dimensional addition ($\oplus$), subtraction ($\ominus$), multiplication ($\odot$), division ($\oslash$), and equality ($\doteq$). For notational conciseness, we will also use an exponentiation operator, exp.

3.1 Laws of dimensional arithmetic, revisited

Let U_1, U_2, and U be any units. Then

1. $U_1 \oplus U_2 \doteq U \leftrightarrow U_1 \doteq U_2 \doteq U$

2. $U_1 \ominus U_2 \doteq U \leftrightarrow U_1 \doteq U_2 \doteq U$

3. $U \otimes u_0 \doteq u_0 \otimes U \doteq U$

4. $U \oslash u_0 \doteq U$

5. $U_1 \otimes U_2 \doteq U_1 * U_2$

6. $U_1 \oslash U_2 \doteq U_1 / U_2$

7. $exp(U, n) \doteq U^n \quad \forall n$

8. $u_0^n \doteq u_0 \quad \forall n$

9. $f(u_0) \doteq u_0$ for any other function f.

It is easily seen that these laws meet the requirements for dimensional arithmetic as stated in §2.2. Note that, except for verification of dimensional equality, this is now a problem in numeric arithmetic. These laws of arithmetic and the laws of dimensional consistency are combined to obtain the following result.[5]

Proposition 1

Any unit term can be represented as $\prod_i u_i^{\alpha_i}$ $(i \geq 0)$ where

$$\underbrace{u_i^{\alpha_i} = u_i \otimes \ldots \otimes u_i}_{\alpha_i \; times} \quad \text{and}$$

$$\prod_{i \geq 0} u_i^{\alpha_i} = u_0^{\alpha_0} \otimes u_1^{\alpha_1} \otimes u_2^{\alpha_2} \ldots$$

---

[5]It follows as well by conceptualizing systems of units as finite Abelian groups [11].

with a suitable choice of α_is ($\in Z$). (For example, power—a derived quantity—has units $kg^1 \otimes m^2 \otimes s^{-3}$)

It is easy to see, given our laws, that this is indeed the case. Assume it were not. Then two possibilities exist. The first is that there is some unit term of the form $\phi \oplus \psi$ or $\phi \ominus \psi$, where ϕ and ψ are in the correct form. However, by law 1 of dimensional consistency, it follows that $\phi \doteq \psi$, else we have an invalid expression. But by laws 1 and 2 of dimensional arithmetic, it follows that this unit term collapses to ϕ which is in the required form. The second possibility is that some α_i is non-integral. But that violates law 5 (for exponents) of dimensional consistency. Hence the stated proposition is true. $\Diamond$.

3.2 Prime-encoding of Dimensions

Proposition 1 motivated the encoding of units of measurement as prime numbers. We first specify this encoding and then discuss its implications. Let $\{p_0, p_1, p_2, \ldots\}$ be a sequence of the successive prime numbers, with $p_0 = 1$. Thus $p_1 = 2$, $p_2 = 3$, $p_5 = 11$, and so on. Let $u_0, u_1, \ldots$ be the various fundamental units of measurement with $u_0 = 1$, and the others in any arbitrary order. We encode these units as prime numbers by substituting p_i for u_i for all i.

Combining Proposition 1 and this encoding, it follows that any unit term can be represented in what we will call the *general exponents form* as

$$\prod_{i \geq 0} p_i^{\alpha_i}, \quad \alpha_i \in Z \quad \forall i$$

Further, for our purposes it is useful to consider another form, which we call the *positive exponents form*, in which each unit term x is represented as

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$$\frac{\displaystyle\prod_{i>0,\gamma_i>0} p_i^{\gamma_i}}{\displaystyle\prod_{i>0,\delta_i>0} p_i^{\delta_i}} = \frac{\mu(x)}{\nu(x)}$$

This is easily obtained from the first form by collecting all the numbers with negative exponents in the denominator and reversing the sign, and eliminating those with exponent zero. $\mu(x)$ and $\nu(x)$ are the arithmetic products of the numbers in the numerator and denominator respectively. It is equally simple to do the reverse transformation. We find it useful to employ the first form for our proofs, and the second one in our computations.

3.3 Law of Dimensional Equivalency

First we recall the unique factorization theorem (also called the fundamental theorem of arithmetic, [21]).

Theorem 1 *Unique Factorization Theorem*

Let a and b be numbers with prime-power factorizations

$$a = \prod_{i=0}^{k} p_i^{\alpha_i} \qquad b = \prod_{i=0}^{k} p_i^{\beta_i}$$

Then $a = b$ iff $\alpha_i = \beta_i$ for all i.

Our second theorem establishes the law for dimensional equivalency.

Theorem 2 *Dimensional Equality $\equiv$ Numeric Equality*

Let x and y be any two unit terms in exponent form (either one). Then $x \doteq y \equiv x = y$. That is, dimensional equality is equivalent to numeric equality when x and y are expressed as products of primes.

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Proof

Consider x and y in the general exponent form. Let $x = \prod_i p_i^{\gamma_i}$ and $y = \prod_i p_i^{\delta_i}$. Then $x = y$ implies that $\gamma_i = \delta_i$ for all i (theorem 1). This in turn implies (by construction) that x and y have exactly the same fundamental units, each occurring with the same exponent. Hence they are dimensionally equal. The converse is obvious, i.e., dimensional equality implies numeric equality.

Since the problem of dimensional arithmetic was earlier reduced to that of verifying dimensional equality and dimensional simplification, we now have everything that we needed. The laws for dimensional arithmetic, as discussed in §3.1, can be directly implemented since dimensional equality is easily tested. Multiplication of two unit terms simply involves multiplication of the $\mu's$ and the $\nu's$ and division requires cross multiplication of $\mu's$ and $\nu's$. We thus have a simple algorithm for doing dimensional manipulation.

1. Devise a prime-encoding for the fundamental units in the system.

2. Treat u_0 as 1, an identity for multiplication.

3. Compute the dimension of any functional expression in positive exponents form by performing multiplication, division, addition, subtraction, and exponentiation as stated below.

4. For dimensional equality of two dimensions, cross-multiply their $\mu's$ and $\nu's$ and check for numeric equality.

5. Treat dimensional multiplication, division, and exponentiation as numeric multiplication, division, and exponentiation respectively.

6. For dimensional addition and subtraction, check dimensional equivalency.

4 Dimension Representation and Manipulation

In this section we discuss a generic logic-based language, L_{dim}, for representing dimensional information and for rules for manipulating this information (e.g., rules for transformation of units. (We will present L_{dim} only in part, a part that meets our requirements in this paper.)

4.1 L-dim

L_{dim} is a specialized language of first-order logic. It is defined in terms of the sets $\mathcal{C}$ (individual constants), $\mathcal{F}$ (function constants), and $\mathcal{R}$ (relation constants), discussed below.

- $\mathcal{C}$: There is a countable number of individual constants in L_{dim}. These include

    - a countable number of **Systems of Measurement** (e.g., **SI**, **English**) which we denote by the symbols
      $s_0, s_1, \ldots$
    - a countable number of **Quantities**, which we denote by the symbols
      $q_0, q_1, q_2, \ldots$
      For example, these are Length, Mass, Time, Area, and so on.
    - a countable number of **Units**, which we denote by the symbols
      $u_0, u_1, u_2, \ldots$
      For example, these are Meter, Kilogram, Feet, Foot-Pounds, and so on. By convention, dimensionless variables have unit $u_0 = 1$.
    - finite-precision **numbers** $n_0, n_1, n_2, \ldots$
      Thus the individual constants in L_{dim} are
      $s_0, s_1, \qquad q_0, q_1, q_2, \ldots, u_0, u_1, u_2, \ldots, n_0, n_1, n_2, \ldots$

- $\mathcal{F}$: The function constants in L_{dim} are:
  $\oplus, \ominus, \oslash, \oslash, exp \qquad +, -, *, /, \hat{}$
  The first five are interpreted as the symbols for dimensional arithmetic, according to laws of dimensional arithmetic. The last five have the usual arithmetic interpretation.

- $\mathcal{R}$: The relation constants in L_{dim} are:

  $\doteq, = \quad$ `base-unit, unit`

  The first two are interpreted as the symbols for dimensional and numeric equality respectively. The interpretations for the other relations will be explained shortly.

The well-formed terms of L_{dim} include the following:

1. Any fundamental unit is a term, called a unit-term. Any number is a term.

2. If U_i and U_j are unit-terms, so are $U_i \oplus U_j$ and $U_i \ominus U_j$, $U_i \odot U_j$, and $U_i \oslash U_j$.

3. If ϕ_1, ϕ_2 are any terms, so are $\phi_1 + \phi_2$, $\phi_1 - \phi_2$, $\phi_1 * \phi_2$, and ϕ_1/ϕ_2. For example, $(m \oslash (s \odot s))$, and $((9/5) * {}^\circ C + 32)$ are terms.

4. Nothing else, not allowed by the above, is a term.

The well-formed formulas (wff's) of L_{dim} include the following:

1. If ϕ_2 and ϕ_2 are terms, then $\phi_1 = \phi_2$ is a wff.
   The first rule enables us to declare and infer laws of unit conversion in L_{dim}. For example, $({}^\circ F = (9/5) * {}^\circ C + 32)$, $(m = 0.3048 * ft)$, and $(ft = 12 * in)$ are wff's.

2. If ϕ_1 and ϕ_2 are unit-terms, then $\phi_1 \doteq \phi_2$ is a wff.
   This rule is concerned with dimensional equality. For example, $(m \oslash (s \odot s) \doteq (m \oslash s) \oslash s))$ is a wff.

3. If s_i is a system of measurement, q_i is a quantity, u_i is any unit, then $\texttt{unit}(s_i, q_i, u_i)$ is a wff, meaning that u_i is one of the units used for quantity q_i in system of measurement s_i.
   This rule allows us to declare facts relating units and quantities to systems of measurement. For example, `unit(SI,Length,m)`, `unit(English,Length,ft)`, and `unit(English,Power,ft-lb)` are wff's.

4. If s_i is a system of measurement, q_i is a quantity, u_i is a base-unit, and n_i is a number, then $\texttt{base-unit}(s_i, q_i, u_i, n_i)$ is a wff, meaning that u_i is a base-unit for quantity q_i in system of measurement s_i and is prime-encoded with the number n_i.
   This rule allows us to declare facts relating base units, their prime-encodings,

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and quantities to systems of measurement. For example, `base-unit(SI,Length,m,2)`, `base-unit(English,Length,ft,3)`, and `base-unit(SI,Mass,kg,5)` are wff's.

5. Nothing else, not allowed by the above, is a wff.

Of course, not all well-formed terms and formulas are meaningful or true. For example, $km = 10 * m$, unit(SI,Length,kg), and $m \doteq cm$ are wff's, but are not true under the intended interpretation. What matters is that the required dimensional information can be represented in L_{dim}. The laws for conversion of units are well known and can be stated in L_{dim}, as shown above. We need to state as facts *only* the laws a) for transforming between the **base** units used in different systems of measurement for the same fundamental dimension, and b) for transforming additional units for each dimension in to the **base** unit for that dimension. The remaining transformation rules (e.g., between `cm` and `in`) are inferred by transforming first to the base units (which are determined through the `unit` and `base-unit` declarations) using rules (b), and then using the conversion rules (a) between the base units.

5 Discussion

We have reduced dimensional arithmetic and manipulation to numerical arithmetic by a suitable encoding of fundamental units as primes, and derived units as products of primes. Consequently, and due to theorems 1 and 2, we have a simple and efficient numerical method for dimensional manipulation. Our method behaves as well as a symbolic one in the sense that it retains information about the symbols it is manipulating (we can reconstruct the actual units of measurement from the numeric value of the unit). Being numerical, it is arguably more efficient. Further, it is easy to implement in almost any programming language (unlike implementing symbolic

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dimensional manipulation in, say, Fortran), making it possible to exploit dimensional ananlysis in a wide range of applications. Finally, the approach of reducing this problem to one of numeric arithmetic may be of use elsewhere in symbolic computing.

The method works for mathematical modeling applications where the laws of dimensional consistency, as stated in §2.1, hold. The first four laws are non-controversial. In laws 5–7, the restriction of exponents (to integers, or to fractions of a special type—note, for example, that $\sqrt{a}$, where a measures *Area*, poses no problem) might appear to be unreasonable. However, it finds support in Fleischmann's conceptualization of formulas in a physical system as a multi-dimensional vector space [10], in which any unit term can be written in the form $\prod_i^n B_r^{a_r}$ where B_rs are units, and a_rs are *integers*, and in his result that dimensions form an Abelian group [11]. We do see expressions of the form ϕ^α where α is non-integral, but we argue that these are meaningful only when ϕ is dimenionless, or it has appropriate units such that ϕ^α has non-fractional fundamental units. In other cases the ostensible lack of dimensional consistency arises from the elimintation of certain variables due to simplification. For example, Massey [23] says that

> True, there are careless writers who present terms such as
> ln r, where r represents a radius, that is a length. It will usually
> be found, however, that the ln r has arisen from the integration
> of dr/r and that the writer omitted the integration constant.
> In such a case the ln r term should be $\ln(r/r_0)$, where -ln r
> corresponds to the missing constant.

In practice many models are written in a manner that they might seem to be dimensionally invalid. It seems particularly to be the case in management

science models, where declaration of units, and dimensional analysis, is not common practice. However, a closer examination of the assumptions and conventions often reveals that that is not the case. Vermeulen [27] discussed how indiscriminate use and manipulation of dimensional formulas leads to contradictions (e.g., 1 sec $= 3 * 10^{10}$ cm). He argued that such contradictions can be avoided by following proper conventions in the use of dimensions and the style of writing such formulas, by examining the origin and derivation of the formulas. Perhaps, the enforcement of dimensional validation can aid the understanding of models (and origins of formulas) and improve modeling practice (as well as style and documentation). Try validating, for example,[6] an expression in the well known Wilson's EOQ model (TC is dollars/yr, D is item-units/yr, Q is item-units/order, A is dollars/order, I is a percentage/yr, and C is dollars/item-unit),[7]

$$TC = \frac{A \cdot D}{Q} + \frac{Q}{2} \cdot I \cdot C.$$

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[6]The expression seems dimensionally inconsistent. However, notice that the term $Q/2$ really represents the average inventory, with the inventory having been sampled at 2 points in each order cycle. Thus the units of the average inventory, $Q/2$, are item-units, rather than item-units/order. That, of course, makes the expression dimensionally consistent.

[7]Thanks to Tom Moore for the suggestion.

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